Classical Weyl-Spinor approach to $U(1)$ and non-abelian local gauge theories

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In a previous paper we introduced two linear spinor equations equivalent to the Lorentz Force and stated that these equations were fairly general and could be applied to any force field compatible with Special Relativity. In this paper, via a lagrangian approach, we explore this possibility and obtain classical spinor equations describing the behaviour of fermionic particles not only under an electromagnetic field but also under Yang-Mills and Color fields. We find a covariant derivative defined *along the classical trajectory* of the particle, which can be extended to SU(2) and SU(3) local symmetries, and obtain the Yang-Mills and Color fields in a new classical Weyl-spinor approach to Gauge Theories. In the SU(3) case, the obtained equations which describe the behaviour of quarks under gluon fields could be in principle applied to the quark-gluon plasma phase existing during the first instants of the Universe.

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I. INTRODUCTION

As is well known the lagrangian of a free fermion

$$L = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi,$$

leading to the Dirac equation is not invariant under local phase transformations of the form

$$\psi \to exp\{ie\varphi(x)\}\psi$$
$$\bar{\psi} \to exp\{-ie\varphi(x)\}\bar{\psi}.$$

In order to achieve invariance it is necessary to add a term

$$L_I = e\left(\bar{\psi}\gamma^{\mu}\psi\right)A_{\mu}$$

where the new vector-field A_μ must transform, coupled to the phase-factor, as

$$A_\mu \rightarrow A_\mu + \frac{1}{e}\partial_\mu\varphi,$$

thus local gauge invariance produce the electromagnetic interaction in terms of the potential A_μ. Now, in this work, we would like to ask the question of what are the consequences of applying the gauge principle in the classical spinor formalism emerging from the *Geometrical Principle* introduced in a precedent paper, which shall be referred to as paper I[1]. We shall see here how, in Weyl-Spinor language, local gauge invariance emerges at a deeper level producing the $4 - rank$ spinor describing the electromagnetic field in terms of its physical components (i.e. the electric and the magnectic field strengths). To start with this, we note that, in its more simple form, gauge invariance is already present in the spinor representation of a photon (see paper I), given by a null hermitian spinor (we use Penrose standard notation[2]) as

$$\pi^A\bar{\pi}^{A'},$$

which is manifestly invariant under the transformation

$$\pi^A \rightarrow exp\{ie\varphi(x)\}\pi^A$$
$$\bar{\pi}^{A'} \rightarrow exp\{-ie\varphi(x)\}\bar{\pi}^{A'}.$$

II. THE SPINORIAL LAGRANGIAN

In paper I we obtained the henceforth called *master equations* for a charged particle of mass m and four-momentum

$$p^{AA'} = \frac{1}{\sqrt{2}}\left[\pi^A\bar{\pi}^{A'} + \eta^A\bar{\eta}^{A'}\right]$$

in the presence of an electromagnetic field described by

$$F_{AA'BB'} = \epsilon_{AB}\bar{\phi}_{A'B'} + \epsilon_{A'B'}\phi_{AB}.$$

The equations being:

$$\begin{aligned} \dot{\eta}_A &= -\tfrac{e}{m}\phi_{AB}\eta^B \\ \dot{\pi}_A &= -\tfrac{e}{m}\phi_{AB}\pi^B, \end{aligned} \tag{1}$$

where the dot means derivative with respect to proper time τ (similar expressions for the complex-conjugate spinors hold). These equation were obtained by applying the *Geometrical Principle* to the elements of $S(2, C)$ (i,e., to the individual spinors). However, to give a more complete picture we would like here to re-derive them in analogy with the standard theory, from the *Gauge Principle*. To this end, we start by defining the free action for a particle of mass m as

$$S_f = \frac{1}{m}\int d\tau\dot{\eta}^A\pi_A \tag{2}$$

so that the *minimal action principle*, when applied with respect to η^A

$$\begin{gathered} \delta S_f = 0 = \tfrac{1}{m}\int_{\tau_1}^{\tau_2} d\tau\delta\dot{\eta}^A\pi_A = \\ \tfrac{1}{m}\int_{\tau_1}^{\tau_2} d\tau\left(\tfrac{d}{d\tau}\delta\eta^A\right)\pi_A = \\ \tfrac{1}{m}\int_{\tau_1}^{\tau_2} d\tau\tfrac{d}{d\tau}(\delta\eta^A\pi_A) - \tfrac{1}{m}\int_{\tau_1}^{\tau_2} d\tau\delta\eta^A\dot{\pi}_A, \end{gathered}$$

leads trivially to $\dot{\pi}_A = 0$, since the first integral on the third line of the above equality is null. It is also possible to write $S_f = \frac{1}{m}\int d\tau\dot{\pi}^A\eta_A$ and consider variations of the path with respect

to π^A instead of to η^A. Both possibilities are fully equivalent, since η^A, π^A are constrained by the condition that $\eta^A \pi_A$ is a constant of motion (see Eq. 7 in paper I). Moreover, once we have fixed the components of η^A those of π^A are, up to a constant phase factor, completely determined and viceversa. The lagrangian corresponding to the action (2) is given by

$$L_f = \frac{1}{m} \dot{\eta}^A \pi_A \tag{3}$$

and, the Euler-Lagrange equations

$$\frac{d}{d\tau} \frac{\partial L}{\partial \dot{\eta}^A} - \frac{\partial L}{\partial \eta^A} = 0 \tag{4}$$

when applied to it lead of course to

$$\dot{\pi}_A = 0 \Longrightarrow \pi_A = const. \tag{5}$$

It follows then that $\dot{\eta}^A$ must also be null, since $\dot{\eta}^A \pi_A = \dot{\pi}^A \eta_A$ (this is due to the condition of conserved rest-mass in spinorial representation, given by $\eta^A \pi_A = constant$). We now explore the consequences of local gauge invariance. Let us apply the following transformations (e is the electric charge) *along the classical trajectory of the particle*

$$\begin{array}{c} \eta_A \rightarrow exp\{ie\frac{\varphi(\tau)}{2}\}\eta_A \\ \pi_A \rightarrow exp\{ie\frac{\psi(\tau)}{2}\}\pi_A \end{array} \tag{6}$$

Now, since the condition

$$\frac{d}{d\tau}(\eta_A \pi^A) \rightarrow \frac{d}{d\tau}\left(exp\{ie\frac{\varphi+\psi}{2}\}\eta_A\pi^A\right) = \left[\frac{ie}{2}(\dot{\varphi}+\dot{\psi})\eta_A\pi^A + \frac{d(\eta_A\pi^A)}{d\tau}\right] exp\{ie\frac{\varphi+\psi}{2}\} = 0 \tag{7}$$

must hold, φ and ψ should be related by

$$\dot{\varphi} = -\dot{\psi} \Longrightarrow \varphi = -\psi + const, \tag{8}$$

so, up to a constant phase-factor, the phase shifts of η_A and π_A must be opposite:

$$\begin{aligned} \eta_A &\to exp\{ie\tfrac{\varphi(\tau)}{2}\}\eta_A \\ \pi_A &\to exp\{i\tfrac{k}{2} - ie\tfrac{\varphi(\tau)}{2}\}\pi_A. \end{aligned} \tag{9}$$

With this transformation, the free-lagrangian reads now

$$L_f \to exp\{i\frac{k}{2}\}\frac{1}{m}\left(\frac{ie}{2}\dot{\varphi}\eta^A\pi_A + \dot{\eta}^A\pi_A\right) \tag{10}$$

and, the additional term (neglecting the constant factor) appearing in it can be written as

$$\frac{1}{2m}ie\dot{\varphi}\eta^A\pi_A = \frac{ie}{2m}\dot{\varphi}\epsilon_{AB}\eta^B\pi^A. \tag{11}$$

It is easy to see that, if we add an interaction term of the form

$$-\frac{e}{m^2}\phi_{AB}\eta^B\pi^A \tag{12}$$

and impose the condition to the new field ϕ_{AB} of transforming, under local phase transformations, as[4]

$$\phi_{AB} \to \phi_{AB} + im\frac{\dot{\varphi}}{2}\epsilon_{AB}, \tag{13}$$

then, the new lagrangian

$$L = \frac{1}{m}\dot{\eta}^A\pi_A - \frac{e}{m^2}\phi_{AB}\eta^B\pi^A, \tag{14}$$

is invariant under $U(1)$ local-phase transformations. The transformation that holds for the conjugate second-rank spinor $\bar{\phi}^{A'B'}$, is given by

$$\bar{\phi}_{A'B'} \to \bar{\phi}_{A'B'} - im\frac{\dot{\varphi}}{2}\epsilon_{A'B'}. \tag{15}$$

This kind of transformations leave however invariant the associated four-rank spinor of the Maxwell field strength

$$F_{AA'BB'} = \epsilon_{AB}\bar{\phi}_{A'B'} + \epsilon_{A'B'}\phi_{AB}$$

since, according to (13),(15), transforms as

$$\epsilon_{AB}\bar{\phi}_{A'B'} + \epsilon_{A'B'}\phi_{AB} \to \epsilon_{AB}\bar{\phi}_{A'B'} + \epsilon_{A'B'}\phi_{AB} + im\frac{\dot{\varphi}}{2}\epsilon_{A'B'}\epsilon_{AB} - im\frac{\dot{\varphi}}{2}\epsilon_{AB}\epsilon_{A'B'}, \tag{16}$$

so the physical components of the electromagnetic field are unchanged under $U(1)$ transformations, just as we should expect. On the other hand, in analogy with Classical Mechanics, from the lagrangian (14), we may also define a hamiltonian by setting

$$H = \frac{\partial L}{\partial \dot{\eta}^A}\dot{\eta}^A - L = \frac{\partial L}{\partial \dot{\eta}_A}\dot{\eta}_A - L = \frac{e}{m^2}\phi_{AB}\eta^B\pi^A, \tag{17}$$

since

$$\frac{\partial L}{\partial \dot{\eta}^A} = \frac{1}{m}\pi_A \equiv \hat{\pi}_A, \tag{18}$$

so the conjugate momentum is $p_A = \hat{\pi}_A$ (equivalently $p^A = -\hat{\pi}^A$). The Hamilton equations

$$\frac{\partial H}{\partial \hat{\pi}_A} = \dot{\eta}^A \tag{19}$$

$$\frac{\partial H}{\partial \eta^A} = -\dot{\hat{\pi}}_A \tag{20}$$

then lead to the *master equations*:

$$\dot{\eta}^A = \frac{e}{m}\phi^{AB}\eta_B \tag{21}$$

$$\dot{\pi}_A = -\frac{e}{m}\phi_{AB}\pi^B. \tag{22}$$

Although the spinor hamiltonian given by (17) is entirely classical it can also give discrete values for the energy in appropiate situations. However, as is well-known, the number of these cases is severely constrained by the requirement of Lorentz Covariance. In order to show that equations (1) describe 1/2-spin particles (see also paper I) we shall consider a charged particle of mass m in a constant magnetic field. In the rest frame of the particle the only non-null components of the field spinor ϕ_{AB} are (we take the magnetic field $\vec{B}$ along the z-axis)

$$\phi_{01} = \phi_{10} = -\frac{i}{2} B_0$$

and the individual spinor solutions to the master equations are then

$$\pi^A(\tau) = \sqrt{m} e^{\pm i \frac{\pi}{2}} \begin{pmatrix} e^{-ie \frac{B_0}{2m} \tau} \\ 0 \end{pmatrix}, \quad \eta^A(\tau) = \sqrt{m} \begin{pmatrix} 0 \\ e^{ie \frac{B_0}{2m} \tau} \end{pmatrix}$$

where the global phase factor for the spinor $\eta^A(\tau)$ has been set to unity (this is always possible since solutions are completely determined up to a constant factor). By substitution in the hamiltonian (17) the values of the energies are (the global phase factors of π^A and η^A respectively, in the rest frame, are constrained by the condition that their difference must equal $i\frac{\pi}{2}$ times an odd integer number[1], so two different values of energy are in this case allowed)

$$E = \pm \frac{e}{2m} B_0 .$$

III. COVARIANT DERIVATIVE

For further generalizations to non abelian symmetries it will be useful to define a covariant derivative. To this end, we start with the free Lagrangian

$$L_f = \frac{1}{m}\dot{\eta}^A \pi_A$$

and define a covariant derivative along the *classical trajectory of the particle* (i.e. a total derivative) in the following way

$$\frac{D}{d\tau}\eta^A \equiv \frac{d\eta^A}{d\tau} - \frac{e}{m}\phi^{AB}\eta_B. \tag{23}$$

Now it is easy to check that the interacting Lagrangian obtained in the preceeding *section* can be obtained by merely replacing the ordinary derivative by the covariant one:

$$\begin{gathered} L_{int} = \tfrac{1}{m}\left(\tfrac{D}{d\tau}\eta^A\right)\pi_A \\ \tfrac{1}{m}(\dot{\eta}^A - \tfrac{e}{m}\phi^{AB}\eta_B)\pi_A = \\ \tfrac{1}{m}\dot{\eta}^A\pi_A - \tfrac{e}{m^2}\phi^{AB}\eta_B\pi_A. \end{gathered}$$

But we would like to show that the covariant derivative (23) in this way defined is also a *gauge* derivative which in the $U(1)$ context means that, if under a local change of phase (along the classical trajectory)

$$\eta'_A = exp\{ie\frac{\varphi(\tau)}{2}\}\eta_A$$

then

$$\frac{D'}{d\tau}\eta'_A = exp\{ie\frac{\varphi(\tau)}{2}\}\frac{D}{d\tau}\eta_A.$$

<u>*Proof*</u>

$$\begin{gathered} \tfrac{D'}{d\tau}\eta'_A = \tfrac{d}{d\tau}\left(e^{ie\frac{\varphi(\tau)}{2}}\eta_A\right) + \tfrac{e}{2m}\phi'_{AB}\epsilon^{BC}e^{ie\frac{\varphi(\tau)}{2}}\eta_C = \\ ie\tfrac{\dot{\varphi}}{2}e^{ie\frac{\varphi(\tau)}{2}}\eta_A + e^{ie\frac{\varphi(\tau)}{2}}\dot{\eta}_A + \tfrac{e}{2m}\left(\phi_{AB} + im\dot{\varphi}\epsilon_{AB}\right)\epsilon^{BC}e^{ie\frac{\varphi(\tau)}{2}}\eta_C \end{gathered}$$

where we have made the replacement

$$\phi'_{AB} = \phi_{AB} + im\frac{\dot{\varphi}}{2}\epsilon_{AB}$$

which, as previously obtained, is the transformation rule of ϕ_{AB} under a local phase change so the $\dot{\varphi}$ terms cancel out and we get

$$\frac{D'}{d\tau}\eta'_A = e^{ie\frac{\varphi(\tau)}{2}}\dot{\eta}_A + \frac{e}{2m}\phi_{AB}\epsilon^{BC}e^{ie\frac{\varphi(\tau)}{2}}\eta_C = e^{ie\frac{\varphi(\tau)}{2}}\frac{D}{d\tau}\eta_A.$$

IV. NON ABELIAN SYMMETRIES AND CLASSICAL EQUATIONS OF MOTION

In tensor language, purely classical theories in which is still possible to speak about particle trajectories parametrized by proper time and consistent with special relativity end with electrodynamics. However, in Weyl-spinor language, the classical formalism developed in the precedent section can be extended to describe other interactions as well. Although our aim is to obtain classical equations of motion for quarks and the color fields, for the sake of clarity we shall first consider the local $SU(2)$ symmetry leading to the Yang-Mills fields. Then, the extension to $SU(3)$ is inmediate. To go over this subject first consider the following identities (we adopt the convention that the scalar $\eta^A\pi_A$ is real and equals the rest-mass m of the particle under consideration)

$$\begin{aligned} p^{AA'}\pi_A &= \tfrac{m}{\sqrt{2}}\bar{\eta}^{A'} \\ p_{AA'}\bar{\eta}^{A'} &= \tfrac{m}{\sqrt{2}}\pi_A \end{aligned}$$

which can be summarized into a single expression

$$\gamma^\mu p_\mu \psi = m\psi \tag{24}$$

by means of the bispinor (if we were considering both π^A and η^A as being distributions over the momentum-space we should speak of ψ as a Dirac bispinor. However this is not the

case: η^A, π^A are here defined along the classical trajectory of the particle)

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi_A \\ \bar{\eta}^{A'} \end{pmatrix}$$

being γ^μ the Weyl representation of the well-known *gamma* matrices. Then the Lorentz scalar

$$\bar{\psi}\psi = \psi^\dagger \gamma^0 \psi = \frac{1}{2} \begin{pmatrix} \bar{\pi}_{A'} & \eta^A \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \pi_A \\ \bar{\eta}^{A'} \end{pmatrix} = \eta^A \pi_A = m$$

is invariant under local $U(1)$ transformations

$$\psi \to \psi' = exp\{ie\frac{\varphi(\tau)}{2}\}\psi.$$

A $SU(2)$ transformation of the form

$$\psi \to \psi' = exp\{i\frac{g}{2}\vec{\tau} \cdot \vec{\varphi}(\tau)\}\psi \tag{25}$$

($\vec{\tau}$ is the iso-vector containing the three Pauli matrices) requires however to act upon a two-component bispinor

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

made up from the tensorial product

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi_A \\ \bar{\eta}^{A'} \end{pmatrix} \otimes (\alpha\hat{\xi} + \beta\hat{\eta}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha \begin{pmatrix} \pi_A \\ \bar{\eta}^{A'} \end{pmatrix} \\ \\ \beta \begin{pmatrix} \pi_A \\ \bar{\eta}^{A'} \end{pmatrix} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \begin{pmatrix} \pi_{1A} \\ \bar{\eta}_1^{A'} \end{pmatrix} \\ \\ \begin{pmatrix} \pi_{2A} \\ \bar{\eta}_2^{A'} \end{pmatrix} \end{pmatrix} \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{26}$$

where $\{\hat{\xi},\hat{\eta}\}$ is some orthonormal basis in the new 2-complex-dimensional vector space introduced. Note that via (26) we have defined a new set of four spinors (barred quantities mean complex conjugate)

$$\begin{aligned}\pi_{1A} &= \alpha\pi_A \\ \pi_{2A} &= \beta\pi_A \\ \eta_1^A &= \bar{\alpha}\eta^A \\ \eta_2^A &= \bar{\beta}\eta^A\end{aligned} \tag{27}$$

which englobe the coefficients α,β. A transformation on $\{\alpha,\beta\}$ then induces a transformation on $\{\pi_a^A,\eta_a^A\}$ $(a=1,2)$. According to (26), (24) should now read

$$\begin{pmatrix} \gamma^\mu p_\mu & 0 \\ 0 & \gamma^\mu p_\mu \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = m\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{28}$$

Actually, under (25) (taken as infinitesimal) the coefficients α,β transform as

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \to \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + i\frac{g}{2}\begin{pmatrix} \varphi_3 & \varphi_1 - i\varphi_2 \\ \varphi_1 + i\varphi_2 & -\varphi_3 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

and the induced transformation on $\{\pi_{aA},\eta_a^A\}$ is then given by

$$\begin{pmatrix} \pi_{1A} \\ \pi_{2A} \end{pmatrix} \to \begin{pmatrix} \pi_{1A} \\ \pi_{2A} \end{pmatrix} + i\frac{g}{2}\begin{pmatrix} \varphi_3 & \varphi_1 - i\varphi_2 \\ \varphi_1 + i\varphi_2 & -\varphi_3 \end{pmatrix}\begin{pmatrix} \pi_{1A} \\ \pi_{2A} \end{pmatrix} \tag{29}$$

$$\begin{pmatrix} \eta_1^A \\ \eta_2^A \end{pmatrix} \to \begin{pmatrix} \eta_1^A \\ \eta_2^A \end{pmatrix} - i\frac{g}{2}\begin{pmatrix} \varphi_3 & \varphi_1 + i\varphi_2 \\ \varphi_1 - i\varphi_2 & -\varphi_3 \end{pmatrix}\begin{pmatrix} \eta_1^A \\ \eta_2^A \end{pmatrix}, \tag{30}$$

leaving the scalar (Einstein's summation convention over lower case indexes is understood)

$$\bar{\psi}\psi = \psi^\dagger\gamma^0\psi = Re\{\eta_1^A\pi_{1A} + \eta_A^2\pi_{2A}\} = (|\alpha|^2 + |\beta|^2)\eta^A\pi_A = \eta_a^A\pi_{aA}$$

invariant due to the unitary character of (25). If we want this scalar to equal the rest-mass

m as in the $U(1)$ case, we must impose

$$|\alpha|^2 + |\beta|^2 = 1.$$

We now relate the set of spinors $\{\pi_a^A, \eta_a^A\}$ in (27) to a fermionic particle of mass m and four-momentum $(a, b = 1, 2)$

$$p_{ab}^{AA'} = \frac{1}{\sqrt{2}} \left(\pi_a^A \bar{\pi}_b^{A'} + \eta_a^A \bar{\eta}_b^{A'} \right),$$

consistent with

$$p_{ab}^{AA'} p_{abAA'} = m^2.$$

We are then concerned with spinorial gauge-fields leading to equations of motion for charged particles which remain invariant under (25) and preserve, along the trajectory, the scalar quantity $\eta_a^A \pi_{aA}$ so that

$$\frac{d}{d\tau}(\eta_a^A \pi_{aA}) = 0.$$

Let us now explore the consequences of $SU(2)$ local phase transformations along the classical path of the particle. Note that the relations (29),(30) in matrix-form are equivalent to

$$\pi_{aA} \to \pi'_{aA} = [exp\{\frac{i}{2} g \vec{\tau} \cdot \vec{\varphi}(\tau)\}]_{ab} \pi_{bA}$$
$$\eta_{aA} \to \eta'_{aA} = [exp\{-\frac{i}{2} g \vec{\tau}^* \cdot \vec{\varphi}(\tau)\}]_{ab} \eta_{bA},$$

where $\vec{\tau}^*$ is the transpose iso-vector and g, as usual, a coupling constant. The covariant derivative should then be of the form

$$\frac{D\eta_{aA}}{d\tau} \equiv \frac{d\eta_{aA}}{d\tau} + [\frac{g}{m} \vec{\tau}^* \cdot \vec{\chi}_{AB}]_{ab} \eta_b^B, \tag{31}$$

where $\vec{\chi}_{AB}$ is an iso-spinor-vector that should transform according to the adjoint three dimensional representation of $SU(2)$. In order to find out how $\vec{\chi}_{AB}$ transform we note that from the gauge derivative condition

$$\frac{D'}{d\tau}\eta'_{aA} = [exp\{-\frac{i}{2}g\vec{\tau}^* \cdot \vec{\varphi}(\tau)\}]_{ab}\frac{D}{d\tau}\eta_{bA}$$

and, considering an infinitesimal transformation with parameter $\vec{\alpha}(\tau)$, the left-hand side of the above equation is

$$\begin{array}{c} \frac{D'}{d\tau}\eta'_{aA} = \frac{D'}{d\tau}\left[1 - i\frac{g}{2}\vec{\tau}^* \cdot \vec{\alpha}(\tau)\right]_{ab}\eta_{bA} = \frac{D'}{d\tau}\left[\eta_{aA} - i\frac{g}{2}\left(\vec{\tau}^* \cdot \vec{\alpha}(\tau)\right)_{ab}\eta_{bA}\right] = \\ \frac{d\eta_{aA}}{d\tau} + (\frac{g}{m}\vec{\tau}^* \cdot \vec{\chi}'_{AB})_{ab}\eta^B_b + \frac{d}{d\tau}\left[\left(-\frac{ig}{2}\vec{\tau}^* \cdot \vec{\alpha}(\tau)\right)_{ab}\eta_{bA}\right] + (\frac{g}{m}\vec{\tau}^* \cdot \vec{\chi}'_{AB})_{ab}\left[\frac{-ig}{2}\vec{\tau}^* \cdot \vec{\alpha}(\tau)\right]_{bc}\eta_{cA} \end{array} \tag{32}$$

while the right-hand side equals to

$$\dot{\eta}_{aA} + (\frac{g}{m}\vec{\tau}^* \cdot \vec{\chi}_{AB})_{ab}\eta^B_b - (\frac{i}{2}g\vec{\tau}^* \cdot \vec{\alpha}(\tau))_{ab}\dot{\eta}_{bA} - (\frac{i}{2}g\vec{\tau}^* \cdot \vec{\alpha}(\tau))_{ab}(\frac{g}{m}\vec{\tau}^* \cdot \vec{\chi}_{AB})_{bc}\eta^B_c$$

Since $\vec{\alpha}(\tau)$ is infinitesimal we assume

$$\vec{\chi}_{AB} \rightarrow \vec{\chi}_{AB} + \delta\vec{\chi}_{AB}$$

where $\delta\vec{\chi}_{AB}$ is small. By substitution in (32), after some lengthy calculations and using the well-known relation

$$(\vec{a} \cdot \vec{\tau}^*)(\vec{b} \cdot \vec{\tau}^*) = \vec{a} \cdot \vec{b} + i(\vec{a} \times \vec{b}) \cdot \vec{\tau}^*,$$

we find

$$\vec{\chi}'_{AB} = \vec{\chi}_{AB} - im\dot{\vec{\alpha}}\epsilon_{AB} - g(\vec{\alpha} \times \vec{\chi}_{AB}). \tag{33}$$

To find the spinor-vector fields $\vec{F}_{AA'BB'}$ we note that, since every component must be skew, it is always possible to write

$$\vec{F}_{AA'BB'} = \vec{\chi}_{AB}\epsilon_{A'B'} + \bar{\vec{\chi}}_{A'B'}\epsilon_{AB} \tag{34}$$

where $\bar{\vec{\chi}}_{A'B'}$ is also symmetric and transforms as

$$\bar{\vec{\chi}}'_{A'B'} = \bar{\vec{\chi}}_{A'B'} + im\bar{\vec{\alpha}}\epsilon_{A'B'} - g(\vec{\alpha} \times \bar{\vec{\chi}}_{A'B'}).$$

In turn, $\vec{F}_{AA'BB'}$ should transform as an iso-vector in the following way

$$\vec{F}'_{AA'BB'} = \vec{F}_{AA'BB'} - g(\vec{\alpha} \times \vec{F}_{AA'BB'}) \tag{35}$$

which, as can be easily verified, is indeed the case for $\vec{F}_{AA'BB'}$ given by (34). The above development lead us to the following Lagrangian with interaction term

$$L = \frac{1}{m}\dot{\eta}_a^A \pi_{aA} - \frac{g}{m^2}\left[(\vec{\tau}^* \cdot \vec{\chi}^{AB})\right]_{ab} \eta_{bB}\pi_{aA} \tag{36}$$

The Hamiltonian is then given by

$$H = \frac{\partial L}{\partial \dot{\eta}_a^A}\dot{\eta}_a^A - L = \frac{g}{m^2}\left[(\vec{\tau}^* \cdot \vec{\chi}_{AB})\right]_{ab} \eta_b^B \pi_a^A \tag{37}$$

and equals the interaction term. Once expanded, the term $(\vec{\tau}^* \cdot \vec{\chi}_{AB})$ is

$$\begin{pmatrix} \chi^3_{AB} & \chi^1_{AB} + i\chi^2_{AB} \\ \chi^1_{AB} - i\chi^2_{AB} & -\chi^3_{AB} \end{pmatrix} \equiv \begin{pmatrix} \chi^3_{AB} & \sqrt{2}\chi^-_{AB} \\ \sqrt{2}\chi^+_{AB} & -\chi^3_{AB} \end{pmatrix}$$

with

$$\chi^{\pm}_{AB} = \frac{1}{\sqrt{2}}(\chi^1_{AB} \mp i\chi^2_{AB})$$

so, for a typical interaction term, we have

$$(\vec{\tau}^* \cdot \vec{\chi}_{AB})_{ab}\pi_a^A\eta_b^B = \chi_{AB}^3\pi_1^A\eta_1^B + \sqrt{2}\chi_{AB}^-\pi_2^A\eta_1^B + \chi_{AB}^+\pi_1^A\eta_2^B - \chi_{AB}^3\pi_2^A\eta_2^B .$$

From (36), the associated spinor equations of motion for η_{aA} and π_{aA} respectively are found to be

$$-\frac{1}{m}\dot{\pi}_{aA} = \frac{\partial H}{\partial \eta_a^A} = \frac{g}{m^2}(\vec{\tau}^* \cdot \vec{\chi}_{BA})_{ba}\pi_b^B = \frac{g}{m^2}(\vec{\tau} \cdot \vec{\chi}_{BA})_{ab}\pi_b^B$$
$$\dot{\eta}_{aA} = -m\frac{\partial H}{\partial \pi_a^A} = -\frac{g}{m}(\vec{\tau}^* \cdot \vec{\chi}_{AB})_{ab}\eta_b^B$$

equivalent to

$$\dot{\eta}_{aA} = -\frac{g}{m}(\vec{\tau}^* \cdot \vec{\chi}_{AB})_{ab}\eta_b^B, \tag{38}$$
$$\dot{\pi}_{aA} = -\frac{g}{m}(\vec{\tau} \cdot \vec{\chi}_{AB})_{ab}\pi_b^B. \tag{39}$$

Once expanded, they become

$$\begin{aligned}
\dot{\eta}_A^1 &= -\tfrac{g}{m}\left[\chi_{AB}^3\eta^{1B} + \sqrt{2}\chi_{AB}^-\eta^{2B}\right] \\
\dot{\eta}_A^2 &= -\tfrac{g}{m}\left[\sqrt{2}\chi_{AB}^+\eta^{2B} - \chi_{AB}^3\eta^{1B}\right] \\
\dot{\pi}_A^1 &= -\tfrac{g}{m}\left[\chi_{AB}^3\pi^{1B} + \sqrt{2}\chi_{AB}^+\pi^{2B}\right] \\
\dot{\pi}_A^2 &= -\tfrac{g}{m}\left[\sqrt{2}\chi_{AB}^-\pi^{2B} - \chi_{AB}^3\pi^{1B}\right].
\end{aligned}$$

Note that, if instead of $SU(2)$ we consider the symmetry group $U(1)$ (one single infinitesimal generator equal to unity and, therefore, one single associated spinorial field ϕ^{AB}) then, the equations of motion of section **II** for the electromagnetic field are recovered. Finally, it is convenient to point out that equations (38),(39) give account of a coupling between the orbital $\{\pi^A, \eta^A\}$ degrees of freedom and the internal $\{\alpha, \beta\}$ ones.

V. COLOR QUARK DYNAMICS

Going further, we would like to extend the above development to $SU(3)$ local phase transformations. If we assume, for simplicity, a single flavour of quark, the analysis of the preceeding section can be easily extended to any other symmetry group and, in particular, to describe color quark dynamics in classical Weyl-spinor language. This is done just by substituying the $SU(2)$-infinitesimal generators by those of $SU(3)$ in the expression for the lagrangian (35), so we here shall quote the principal results only. For a single quark of mass m existing in three color charges, the Lagrangian is $(a = 1, 2, 3)$

$$L = \frac{1}{m}\dot{\eta}_a^A \pi_{aA} - \frac{g_s}{m^2}[\lambda^*_{\mathbf{q}} W^{\mathbf{q}}_{AB}]_{ab}\eta_b^B \pi_a^A, \tag{40}$$

where $\lambda^*_{\mathbf{q}}$ $(\mathbf{q} = 1, 2, ..., 8)$ are the transposed Gell-Mann $SU(3)$ matrices and $W^{\mathbf{q}}_{AB}$ the 8 boson spinor fields. The Hamiltonian in this case is given by

$$H = \frac{g_s}{m^2}[\lambda^*_{\mathbf{q}} W^{\mathbf{q}}_{AB}]_{ab}\eta_b^B \pi_a^A, \tag{41}$$

and the gauge fields transform as $(\mathbf{i}, \mathbf{j}, \mathbf{k} = 1, 2, ..., 8)$

$$\begin{aligned} W'^{\mathbf{i}}_{AB} &= W^{\mathbf{i}}_{AB} - im\dot{\alpha}^{\mathbf{i}}\epsilon_{AB} - f^{\mathbf{ijk}}\alpha^{\mathbf{j}}(\tau)W^{\mathbf{k}}_{AB} \\ \bar{W}'^{\mathbf{i}}_{A'B'} &= \bar{W}^{\mathbf{i}}_{A'B'} + im\dot{\alpha}^{\mathbf{i}}\epsilon_{A'B'} - f^{\mathbf{ijk}}\alpha^{\mathbf{j}}(\tau)\bar{W}^{\mathbf{k}}_{A'B'} \end{aligned}$$

where $f^{\mathbf{ijk}}$ are the structure constants of $SU(3)$. For the fourth-rank spinor field one obtains

$$F^{\mathbf{i}}_{AA'BB'} = W^{\mathbf{i}}_{AB}\epsilon_{A'B'} + \bar{W}^{\mathbf{i}}_{A'B'}\epsilon_{AB} \tag{42}$$

and $F_{AA'BB'}$ transforms as

$$F'^{\mathbf{i}}_{AA'BB'} = F^{\mathbf{i}}_{AA'BB'} - g_S\left[f^{\mathbf{ijk}}\alpha^{\mathbf{j}}(\tau)F^{\mathbf{k}}_{AA'BB'}\right].$$

The equations of motion are now given by

$$\dot{\eta}_{aA} = -\frac{g_S}{m}(\lambda^*_{\mathbf{q}} W^{\mathbf{q}}_{AB})_{ab}\eta^B_b, \tag{43}$$

$$\dot{\pi}_{aA} = -\frac{g_S}{m}(\lambda_{\mathbf{q}} W^{\mathbf{q}}_{AB})_{ab}\pi^B_b. \tag{44}$$

In a more explicit form, these equations are

$$\begin{pmatrix} \dot{\eta}^1_A \\ \dot{\eta}^2_A \\ \dot{\eta}^3_A \end{pmatrix} = -\frac{g_S}{m} \begin{pmatrix} W^3_{AB} + \frac{W^8_{AB}}{\sqrt{3}} & W^1_{AB} + iW^2_{AB} & W^4_{AB} + iW^5_{AB} \\ W^1_{AB} - iW^2_{AB} & -W^3_{AB} + \frac{W^8_{AB}}{\sqrt{3}} & W^6_{AB} + iW^7_{AB} \\ W^4_{AB} - iW^5_{AB} & W^6_{AB} - iW^7_{AB} & -\frac{2W^8_{AB}}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \eta^{1B} \\ \eta^{2B} \\ \eta^{3B} \end{pmatrix}. \tag{45}$$

$$\begin{pmatrix} \dot{\pi}^1_A \\ \dot{\pi}^2_A \\ \dot{\pi}^3_A \end{pmatrix} = -\frac{g_S}{m} \begin{pmatrix} W^3_{AB} + \frac{W^8_{AB}}{\sqrt{3}} & W^1_{AB} - iW^2_{AB} & W^4_{AB} - iW^5_{AB} \\ W^1_{AB} + iW^2_{AB} & -W^3_{AB} + \frac{W^8_{AB}}{\sqrt{3}} & W^6_{AB} - iW^7_{AB} \\ W^4_{AB} + iW^5_{AB} & W^6_{AB} + iW^7_{AB} & -\frac{2W^8_{AB}}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \pi^{1B} \\ \pi^{2B} \\ \pi^{3B} \end{pmatrix}. \tag{46}$$

Final Remarks

As already emphasized the Weyl-spinor approach permits an extension of purely classical physics far beyond electrodynamics. On the other hand, it seems now that the spinor *master equations* obtained in paper I can be applied to a wide variety of situations and, perhaps (of special interest), in the cosmological scenario of the primordial quark-gluon plasma state ending in quark confinement. It is curious to realize that equations (45), (46), which describe the dynamical behaviour of quarks in the presence of gluon fields, mimic the Lorentz-force of electrodynamics (although with a much greater level of complexity). This has in fact some experimental support, since it has been already pointed out the similarity between the energy levels of charmonium ($c\bar{c}$) (due to strong forces) and positronium (e^+e^-) (due to electromagnetic forces)[3]. The assertion made in paper I about the fully generality of the *master equations* is then supported by these results.

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[2] R. Penrose and W. Rindler *Spinors and Space-Time*, Cambridge Monographs in Mathematical Physics, Vol. 1, Cambridge Universtiy Press, Cambridge, England (1984/1986).

[3] I. Aitchison and A. Hey *Gauge Theories in Particle Physics* , Institute of Physics Publishing, Vol. 1, 150 South Independence Mall West, Philadelphia, PA 19106, USA (2003).

[4] The validity of transformation (13) is consequence of the following theorem applied to valence-2 spinors (see Steward. J.*Advanced General Relativity*. 1991 Cambridge Univ. Press. Page 69): "Any spinor $\tau_{A...F}$ is the sum of the totally symmetric spinor $\tau_{(A...F)}$ and (outer) products of $\epsilon' s$ with totally symmetric spinors of lower valence"